# Assignment 4

No need to hand in any exercise.

1. A trigonometric polynomial is $p(\cos x, \sin x)$ where $p(x,y)$ is a polynomial in two variables. Its degree is the degree of $p$. For instance, letting $p(x,y) = x^2y - 6xy + 3y - 5$ which is of degree 3, the corresponding trigonometric polynomial is $\cos^2 x \sin x - 6\cos x \sin x + 3\sin x - 5$ . Show that every finite trigonometric series
$$\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$
can be expressed as a trigonometric polynomial of degree $n$ and the converse is true.

2. Show that for two continuous, $2\pi$-periodic functions $f$ and $g$, they are identical if their Fourier series are the same. Hint: Show that $\int_{-\pi}^{\pi}(f-g)(x)p(x)dx = 0$ for all finite trigonometric series.

3. Find the first twenty data for the following sequences and count how many are in the intervals $I_1 = [0, 0.25), I_2 = [0.25, 0.75)$ and $I_3 = [0.5, 1)$ respectively in each case.
$$(a)\ \ \langle n\sqrt{3}\rangle\ , \quad (b)\ \ \langle p_n\sqrt{2}\rangle\ , \quad (c)\ \ \left\langle \frac{(1+\sqrt{5})^n}{2} \right\rangle\ .$$
Here $p_n$ is the $n$-th prime number ($p_1 = 2,\ p_2 = 3$, etc). What conclusion on their distribution can you draw? Try more data if you don't see the trend.

4. The Fibonacci numbers are given by the sequence $\{U_n\}$ satisfying
$$U_{n+1} = U_n + U_{n-1}, \quad U_0 = 2,\ U_1 = 1\ .$$
Show that
$$U_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\ , \quad n \geq 0\ .$$
You may use induction.

5. Prove that the sequence $\{\gamma_n\}$, where $\gamma_n$ is the fractional part of $((1+\sqrt{5})/2)^n, n \geq 1$, is not equidistributed in $[0,1)$.

6. (Optional) Show that for $\sigma \in (0,1)$, the sequence $\{\langle n^\sigma \rangle\}$ is equidistributed in $[0,1)$. Hint: Prove that
$$\sum_{n=1}^{N} e^{2\pi ikn^\sigma} = O(N^\sigma) + O(N^{1-\sigma})$$
by noting
$$\sum_{n=1}^{N} e^{2\pi kin^\sigma} - \int_1^N e^{2\pi ikx^\sigma} dx = O\left(\sum_{n=1}^{N} \frac{1}{n^{1-\sigma}}\right)\ .$$

7. Let $f$ be a piecewise continuous, $2\pi$-periodic function and $\sigma_N f$ its $N$-th Cesá sum. Show that $S_N f(x)$ tends to $f(x)$ when $x$ is a point of continuity of $f$ and it tends to $(f(x^+) + f(x^-))/2$ when $x$ is a jump discontinuity.